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Scalar and matrix Riemann–Hilbert approach to the strong asymptotics of Padé approximants and complex orthogonal polynomials with varying weight ☆

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Abstract

We describe methods for the derivation of strong asymptotics for the denominator polynomials and the remainder of Padé approximants for a Markov function with a complex and varying weight. Two approaches, both based on a Riemann–Hilbert problem, are presented. The first method uses a scalar Riemann–Hilbert boundary value problem on a two-sheeted Riemann surface, the second approach uses a matrix Riemann–Hilbert problem. The result for a varying weight is not with the most general conditions possible, but the loss of generality is compensated by an easier and transparent proof.

Keywords: Strong asymptotics; Padé approximants; Riemann-Hilbert problem

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1. Introduction

Recently, there has been considerable progress in proving strong asymptotics for general orthogonal polynomials using methods of complex analysis and a Riemann–Hilbert boundary value problem (BVP) for matrix analytic functions. This new approach has been used extensively by P. Deift and his collaborators and an important reference is [3]. The matrix Riemann–Hilbert problem for orthogonal polynomials was first formulated by Fokas, Its and Kitaev [8], and analyzed by Bleher and Its [2], Deift et al. [4–6], and Kriecherbauer and McLaughlin [10]. We recommend the exposition of Kuijlaars [11] for an introduction to the use of the Riemann–Hilbert approach for the asymptotic analysis of orthogonal polynomials. In the present paper we explain special versions of BVP based approaches. The roots of these versions lie in the research of rational approximants. The first presentation of a version based on BVP on a Riemann surface was in the paper [13] of Nuttall, although the ideas were in [14] and even in earlier papers of Nuttall (related references can be found in [14], see also the paper of Stahl [16]). A substantial development of the approach based on the BVP on a Riemann surface has been done by Suetin in a recent paper [17], where Nuttall's version of the asymptotic analysis has been extended to Padé approximants for hyperelliptic functions. Also we would like to add to the list of references an older paper [1] where the main ingredient, i.e., a boundary value Riemann problem on a Riemann surface, has been considered in connection with strong asymptotics of orthogonal polynomials.

The present paper is intended to describe the Riemann–Hilbert boundary value problem and its relation to strong asymptotics of Padé approximants in a rather general setting. In the appendix we give an introduction to the Riemann boundary value problem. So a reader not familiar with the Riemann boundary value problem is encouraged to start reading the paper from the end. The aim of this appendix is to provide a 'shortcut' in complex analysis to give an idea of how to find a solution of a boundary value problem on a Riemann surface, based just on the notion of Cauchy's integral formula and its generalization, the Cauchy residue theorem. The Riemann surfaces, which will be used in this paper, have genus zero, which allows us to avoid the non-trivial part of the theory of Riemann surfaces and requires from the reader just a 'naive' understanding of a Riemann surface as a two-dimensional manifold in four-dimensional space.

In Section 2 we solve a special Riemann problem, which will be used for the derivation of the strong asymptotics. Then in Section 3, we prove strong asymptotics of Padé approximants for a Markov function generated by an analytic, complex valued weight function, i.e., strong asymptotics for polynomials orthogonal with respect to an analytic complex valued weight. This result is just a repetition of the corresponding theorem of Nuttall from [13], using for its proof ideas and details from Suetin's paper [17]. Nevertheless, there are some new methodological insights which make the proof easier and which indicate that the method can be developed for a wider class of applications. Next in Section 4 we prove a generalization of Nuttall's theorem when the complex valued weight has a varying real valued component. An equilibrium problem in the presence of an external field plays an important role here. Finally (see Section 5), we consider Deift's (with co-authors) version of the Riemann–Hilbert problem approach to the strong asymptotics for orthogonal polynomials (see [3]). The starting point of the approach is a reformulation of the orthogonality relations in terms of a matrix Riemann–Hilbert problem which has been introduced and developed in [7,2]. We choose for this presentation of the approach a model problem of asymptotics for polynomials orthogonal on $[-1,1]$ with respect to a complex weight function—the same problem as in Section 3. Here (again as in Section 3) we assume that the analytic weight function has the same behavior at the end points of the interval of orthogonality as the classical Chebyshev weight. This allows us to get a very easy proof (in the framework of Deift's approach) for the asymptotic formulas. This shortcut avoids most of one delightful ingredient of the matrix Riemann–Hilbert problem approach, which is the analysis around the end points.

Concluding the introduction we have to say that the paper has more emphasis on explaining a method than on exposition of new results. We pay more attention to showing the different points of view on the same subject and to transparency of the proofs than to generality of the results proven here.

2. Auxiliary BVP

In this section, we start from the preliminary material presented in the appendix (see below) to study a function which will be used in the formulation of the asymptotic formulas later on. This function is introduced as a solution of some boundary value problem. The function contains a generalisation of the well known Szegő function for the complex weight.

2.1. Statement of the problem, properties of the solution

Let $\varrho(z)$ be a complex valued function, $\varrho(x)\neq 0$ for $x\in[-1,1]$, and assume that ϱ has an analytic continuation in the domain $\delta\supset[-1,1]$, so that $\varrho\in H(\delta)$. We consider the following BVP:

$$\text{Find } \varphi,\psi \text{ such that} \begin{cases} 1.\ \psi,\varphi\in H(\mathbb{C}\setminus[-1,1]); & \begin{cases} \psi(z)|_\infty = O(z^{-n}), \\ \varphi(z)|_\infty = O(z^{n}), \end{cases} \\ 2.\ \begin{cases} \varrho\varphi_+ = \psi_- \\ \varrho\varphi_- = \psi_+ \end{cases} & \text{on } [-1,1], \\ 3.\ \psi\varphi|_\infty = 1. \end{cases} \tag{1}$$

The boundary values $\varphi_\pm$ and $\psi_\pm$ of φ and ψ are assumed to be uniformly bounded on $[-1,1]$.

We highlight several properties of the solutions of problem (1):

Property 1. *For every* $z \in \overline{\mathbb{C}}$ *we have*

$$\psi(z)\varphi(z) = 1. \tag{2}$$

Proof. From 2 in (1) it follows that $(\varphi\psi)_+ = (\varphi\psi)_-$ on $[-1,1]$, so that 1 in (1) implies that $\varphi\psi$ is analytic in $\overline{\mathbb{C}}$. The maximum principle (or Liouville's theorem) then shows that $\varphi\psi$ is constant, and by 3 of (1) we see that the constant is 1. □

From Property 1 we also have

$$\varrho\varphi_+\varphi_- = 1 \quad \text{on } [-1,1]. \tag{3}$$

Property 2. *We have*

$$\psi, \varphi \neq 0 \ \textit{in} \ \mathbb{C}. \tag{4}$$

Proof. This is an immediate consequence of 1 in (1) and Property 1. □

Let us consider a piecewise analytic function $\mathcal{F}$ on the two-sheeted Riemann surface $\mathcal{R}$ (see Appendix, Fig. 4) consisting of two analytic pieces φ and ψ placed on the sheets $\mathcal{R}^{(+)}$ and $\mathcal{R}^{(-)}$, respectively:

$$\mathcal{F}(z) = \begin{cases} \varphi(z), & z \in \mathcal{R}^{(+)}, \\ \psi(z), & z \in \mathcal{R}^{(-)}. \end{cases} \tag{5}$$

Let Δ be a contour on the Riemann surface splitting $\mathcal{R}$ into two pieces $\{\mathcal{R}^{(+)}\backslash[-1,1]\}$ and $\{\mathcal{R}^{(-)}\backslash[-1,1]\}$. For Δ we can choose a Jordan contour which goes along the upper side of $[-1,1]$ on $\{\mathcal{R}^{(+)}\}$ and along the lower side of $[-1,1]$ on $\{\mathcal{R}^{(+)}\}$ in the clockwise direction. Then the BVP (1) for the functions φ and ψ is equivalent to the BVP on $\mathcal{R}$ for $\mathcal{F}$ given by (5)

$$\text{Find } \mathcal{F} \text{ such that} \begin{cases} 1. \ \mathcal{F} \in H(\mathcal{R}\backslash\{\Delta, \infty^+\}), \quad \mathcal{F}^{(\pm)}(z)|_\infty = O(z^{\pm n}), \\ 2. \ \varrho\mathcal{F}_+ = \mathcal{F}_- \quad \text{on } \Delta, \\ 3. \ \mathcal{F}^{(+)}\mathcal{F}^{(-)}|_\infty = 1. \end{cases} \tag{6}$$

Properties 1 and 2 for (φ, ψ) transform into (see (2) and (3))

$$\mathcal{F}(z^{(+)})\mathcal{F}(z^{(-)}) = 1 \quad \forall z \in \mathbb{C},$$

and

$$\mathcal{F} \neq 0 \quad \text{on } \mathcal{R}.$$

Finally, we mention the uniqueness property of the solution of (1) and (6).

Property 3. *If the solution of the BVP* (6) *exists, then it is unique.*

Proof. Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be two solutions of (6). Then 2 of (6) shows that $\left(\frac{\widetilde{\mathcal{F}}}{\mathcal{F}}\right)_+ = \left(\frac{\widetilde{\mathcal{F}}}{\mathcal{F}}\right)_-$ on Δ, and hence $\frac{\widetilde{\mathcal{F}}}{\mathcal{F}}$ is holomorphic on the whole Riemann surface $\mathcal{R}$, and by the maximum principle on $\mathcal{R}$ we have that $\frac{\widetilde{\mathcal{F}}}{\mathcal{F}} = \text{const}$, and by 3 of (6) we conclude that $\frac{\widetilde{\mathcal{F}}}{\mathcal{F}} = 1$. $\square$

Now we construct a solution of the BVP (6). First we proceed with a special case and after that we consider the general case.

2.2. Special case. Representation of the mapping function and Szegő function for a Bernstein weight by rational functions on $\mathcal{R}$

Let the multiplicative jump in the BVP (6) be given by

$$\varrho := \frac{1}{p},$$

where p is an arbitrary polynomial of degree m with complex coefficients. Polynomials orthogonal on $[-1, 1]$ with respect to the weight $1/(p(x)\sqrt{1-x^2})$ have been considered by S.N. Bernstein, see, e.g., [18, Section 2.6].

Let $\mathcal{B}(z)$ be a function meromorphic on $\mathcal{R}$, which is (like a rational function on $\overline{\mathbb{C}}$) defined by its zeros and poles (divisors) as follows (see Fig. 1):

$$\mathcal{B}: \begin{cases} \bullet \text{ On } \mathcal{R}^{(+)} \text{ at } \infty^{(+)} \text{ there is a pole of order } n, \\ \bullet \text{ On } \mathcal{R}^{(-)} \text{ we have the } m \text{ zeros of } p(z) = c_m z^m + \cdots, \\ \quad \text{and at } \infty^{(-)} \text{ there is a pole of order } m-n. \end{cases}$$

The normalization of $\mathcal{B}$ is chosen as

$$\lim_{z\to\infty} \frac{\mathcal{B}^{(+)}(z)\mathcal{B}^{(-)}(z)}{z^m} = c_m.$$

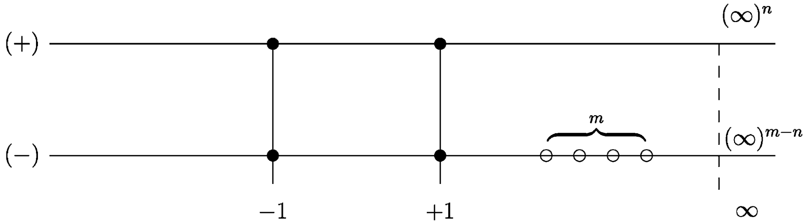

Fig. 1. Zeros and poles for the function $\mathcal{B}$.

It is easy to check that the function

$$\mathcal{F} := \begin{cases} \mathcal{B}^{(+)} & \text{on } \mathcal{R}^{(+)}, \\ \varrho\mathcal{B}^{(-)} & \text{on } \mathcal{R}^{(-)}, \end{cases}$$

satisfies the BVP (6) for $\varrho = 1/p$. Indeed, condition 1 of (6) is valid by construction of $\mathcal{B}$. Condition 2 becomes

$$\varrho\mathcal{B}_+ = \varrho\mathcal{B}_- \quad \text{on } \Delta$$

and it is valid since $\mathcal{B}^{(-)}$ is an analytic continuation of $\mathcal{B}^{(+)}$. Condition 3 is valid because of the normalization of $\mathcal{B}$ at ∞.

The function $\mathcal{B}$ can be decomposed into the product of two rational functions on $\mathcal{R}$

$$\mathcal{B} = F_B\Phi^n,$$

where the zeros of F_B are on the sheet $\mathcal{R}^{(-)}$ at the zeros of p and F_B has a pole of order m at $\infty^{(-)}$ (Fig. 2).

The function Φ has just a simple pole at $\infty^{(+)}$ and a simple zero at $\infty^{(-)}$ (Fig. 3) The normalization is $\Phi^{(+)}\Phi^{(-)}|_\infty = 1$. For the function Φ we know the explicit expression

$$\Phi(z) = z + \sqrt{z^2-1}, \quad \Phi^{(+)}|_\infty = 2z + \cdots. \tag{7}$$

This is the inverse Zhukovsky function: $\Phi(z)$ maps the following regions to the inside and outside of the unit circle:

$$\Phi^{(-)} : \{\overline{\mathbb{C}}\backslash[-1,1]\} \to \{|w| < 1\} =: U,$$

$$\Phi^{(+)} : \{\overline{\mathbb{C}}\backslash[-1,1]\} \to \{|w| > 1\} = \overline{\mathbb{C}}\backslash\bar{U}.$$

2.3. *General case. Determination of the Szegő function for a complex valued weight by means of the solution of the Riemann problem on $\mathcal{R}$*

First we normalize the solution of the BVP (6) on $\mathcal{R}$:

$$F := \frac{\mathcal{F}}{\Phi^n} \quad \text{on } \mathcal{R}.$$

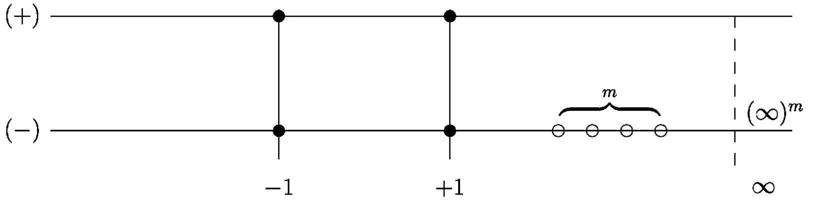

Fig. 2. Zeros and poles for the function F_B.

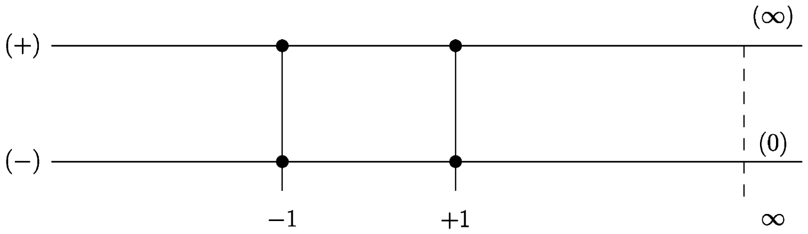

Fig. 3. Zeros and poles for the function Φ.

For the bounded function F on $\mathcal{R}$ (taking into account the properties of $\mathcal{F}$ and Φ) we have

$$\begin{cases} 1.\ F \in H(\mathcal{R}\backslash\Delta);\ \ F \neq 0 \quad \text{on } \mathcal{R}, \\ 2.\ \varrho F_+ = F_- \quad \text{on } \Delta \subset \mathcal{R}, \\ 3.\ F^{(+)}F^{(-)} = 1 \quad \text{on } \overline{\mathbb{C}}. \end{cases}$$

Observe the following equality, which follows from 2 and 3 of the above relation:

$$\varrho F_+^{(+)} F_-^{(+)} = 1 \quad \text{on } [-1, 1] \subset \overline{\mathbb{C}}. \tag{8}$$

Since F does not vanish on $\mathcal{R}$ and $\varrho \neq 0$ on Δ, we can choose a single-valued branch of the complex logarithm, connecting the functions $\ln F^{(+)}$, $\ln F^{(-)}$ and $\ln \varrho$ on Δ. Thus, for the single-valued function $\ln F$ on $\mathcal{R}$, we have the standard Riemann problem (see Appendix (A.7)) on $\mathcal{R}$:

$$\begin{cases} 1.\ \ln F \in H(\mathcal{R}\backslash\Delta), \\ 2.\ (\ln F)_+ - (\ln F)_- = -\ln \varrho \ \text{ on } \Delta, \\ 3.\ \ln F(z^{(+)}) = -\ln F(z^{(-)}), \end{cases}$$

which can be solved using the Cauchy integral (A.8)

$$\ln F(z^{(+)}) = -\frac{1}{2\pi i}\int_\Delta \ln \varrho(\zeta)\, d\Omega(\zeta; z^{(+)}, z^{(-)}) + \ln F(z^{(-)}).$$

Finally, substituting the explicit expression for the meromorphic differential (A.9), we have

$$\begin{aligned} \ln F(z^{(+)}) &= -\frac{1}{4\pi i}\int_\Delta \ln \varrho(\zeta)\, d\Omega(\zeta; z^{(+)}, z^{(-)}) \\ &= -\frac{1}{4\pi i}\int_\Delta \frac{\sqrt{z^2-1}}{\sqrt{\zeta^2-1}}\frac{1}{\zeta - z}\ln \varrho(\zeta)\, d\zeta. \end{aligned}$$

Let us denote by $w(z)$ a branch of $\sqrt{z^2-1}$ on $\mathbb{C}\backslash[-1, 1]$ such that

$$w(x) = (x^2-1)^{\frac{1}{2}} > 0 \quad \text{for } x \in (1, \infty]. \tag{9}$$

Then we have

$$\begin{aligned}\ln F^{(+)}(z) &= \frac{1}{4\pi i}\int_{\Delta} \frac{w(z)}{\sqrt{\zeta^2-1}}\frac{1}{z-\zeta}\ln \varrho(\zeta)\,d\zeta \\ &= \frac{1}{4\pi i}\int_{-1}^{1} \frac{w(z)}{w(x)_+}\frac{\ln \varrho(x)}{z-x}\,dx + \frac{1}{4\pi i}\int_{-1}^{1} \frac{w(z)}{w(x)_-}\frac{\ln \varrho(x)}{z-x}\,dx \\ &= \frac{1}{2\pi i}\int_{-1}^{1} \frac{w(z)}{w(x)_+}\frac{\ln \varrho(x)}{z-x}\,dx.\end{aligned}$$

The function

$$F^{(+)}(z) = \exp\left\{-\frac{1}{2\pi}\sqrt{z^2-1}\int_{-1}^{1}\frac{\ln \varrho(x)}{z-x}\frac{dx}{\sqrt{1-x^2}}\right\} \tag{10}$$

is the reciprocal of the so-called Szegő function, which satisfies the boundary condition (8) and is normalized at infinity as

$$F^{(+)}(\infty) = \exp\left\{-\frac{1}{2\pi}\int_{-1}^{1}\frac{\ln \varrho(x)\,dx}{\sqrt{1-x^2}}\right\}.$$

Thus we obtained

$$\varphi(z) = \Phi^n(z)F^{(+)}(z), \quad \varphi(z)|_{\infty} = F^{(+)}(\infty)(2z)^n + \cdots, \tag{11}$$

where Φ and F are given by (7) and (10).

3. Strong asymptotics for Padé approximants of a Markov function with complex weight

3.1. Jump condition for Padé approximants

We consider a *Markov function*

$$\widehat{\varrho}(z) := \frac{1}{2\pi}\int_{-1}^{1}\frac{\varrho(x)}{z-x}\frac{dx}{\sqrt{1-x^2}} \tag{12}$$

and its *Padé approximants*: a polynomial denominator Q, a polynomial numerator P, and the remainder function R such that

$$\begin{aligned}&1.\ \deg Q,\ \deg P \leqslant n, \quad R|_{z\to\infty} = O\left(\frac{1}{z^{n+1}}\right), \\ &2.\ Q\widehat{\varrho} + P = R.\end{aligned} \tag{13}$$

We assume that ϱ satisfies the conditions of Section 2. Also, for the moment, we will assume that the complex-valued function ϱ is such that $\deg Q = n$. Later on we will show that this is always the case for large enough n. This was already proved by Magnus [12], who also proved the convergence of the Padé approximants using Toeplitz matrix techniques. We take the normalization of Q as

$$Q(z) = z^n + \cdots.$$

From the definition of a Markov function (12), we have by the Sokhotsky–Plemelj (or Stieltjes–Perron) formula

$$\widehat{\varrho}_+ - \widehat{\varrho}_- = \frac{\varrho}{w_+} \quad \text{on } [-1,1],$$

which gives us the jump condition for the remainder function (this follows from definition (13))

$$R_+ - R_- = Q\frac{\varrho}{w_+} \quad \text{on } [-1,1]. \tag{14}$$

3.2. Riemann problem on $\mathcal{R}$ for Padé approximants and its solution

We have from (14) and (9)

$$(wR)_+ - Q\varrho = -(wR)_- \quad \text{on } [-1,1]. \tag{15}$$

The idea of what follows is: using the decomposition of ϱ by means of the auxiliary BVP considered in Section 2, we rewrite the above jump condition as a jump for functions analytic in $\overline{\mathbb{C}}\setminus[-1,1]$, for which the Riemann problem on $\mathcal{R}$ can be stated and solved, giving as a result an integral equation for the remainder function R. The analysis of this integral equation leads to the asymptotics for Q and R. So, (15) and (1) give us on $[-1,1]$

$$\varrho = \frac{\psi_-}{\varphi_+} \Rightarrow (wR\varphi)_+ - (Q\psi)_- = -\varphi_+(wR)_-$$

and

$$\varrho = \frac{\psi_+}{\varphi_-} \Rightarrow (wR\varphi)_- - (Q\psi)_+ = -\varphi_-(wR)_+.$$

Thus, for the functions

$$(wR\varphi), (Q\psi) \in H(\overline{\mathbb{C}}\setminus[-1,1]),$$

we have the following jump condition:

$$(wR\varphi)_\pm - (Q\psi)_\mp = \left(-\frac{\psi wR}{\varrho}\right)_\mp.$$

We define on $\mathcal{R}$

$$f(z) := \begin{cases} w(z)R(z)\varphi(z), & z \in \mathcal{R}^{(+)}, \\ Q(z)\psi(z), & z \in \mathcal{R}^{(-)}. \end{cases}$$

Then for f we have the following Riemann problem (see (A.7)) on $\mathcal{R}$:

$$\begin{cases} 1.\ f \in H(\mathcal{R}\setminus\Delta), \\ 2.\ f_+ - f_- = j_- \quad \text{on } \Delta := \{[-1,1]_+ \cup [-1,1]_-\} \subset \mathcal{R}^{(+)}, \\ 3.\ f(\infty^{(-)}) = c_n, \end{cases} \tag{16}$$

where

$$j = -\left(\frac{\psi w R}{\varrho}\right) \quad \text{on } \mathcal{R}^{(-)} \cap \pi^{-1}(\delta)$$

and

$$c_n = \frac{F^{(-)}(\infty)}{2^n} = \frac{1}{2^n} \exp\left\{\frac{1}{2\pi} \int_{-1}^{1} \frac{\ln \varrho(x)\, dx}{\sqrt{1-x^2}}\right\}.$$

The solution of this problem (see (A.8)) is

$$f(z) = \frac{1}{2\pi i} \int_{\Delta} j(\zeta)\, d\Omega(\zeta; z, \infty^{(-)}) + c_n, \tag{17}$$

where the explicit expression for the differential $d\Omega$ is given in (A.12). If we consider (17) for $z \in \mathcal{R}^{(+)}$, we get an integral equation for R.

Now an important point! Taking into account that the jump function $j(\zeta)$ has an analytic continuation from Δ to the $\mathcal{R}^{(-)}$ sheet, we can deform the contour Δ to the contour $\Delta' = \{z : |\Phi(z)|^2 = 1 - \varepsilon, \varepsilon > 0\} \subset \mathcal{R}^{(-)} \cap \pi^{-1}(\delta)$, and for z outside a ring $\mathcal{A}$, bounded by Δ and Δ', we have (see Remark A.1 in Appendix, Section A.1)

$$f(z) = \frac{1}{2\pi i} \int_{\Delta'} j(\zeta)\, d\Omega(\zeta; z, \infty^{(-)}) + c_n, \quad z \in \mathcal{R} \backslash \mathcal{A}. \tag{18}$$

3.3. *Outer asymptotics for Q*

First we estimate $||R||_{C(\Delta)}$. We denote $M_n := ||wR\varphi||_{C(\Delta)}$. Consider Eq. (18) for $z \in \mathcal{R}^{(+)} \backslash \Delta$

$$(wR\varphi)(z) = -\frac{1}{2\pi i} \int_{\Delta'} \left(\frac{w\psi R}{\varrho}\right)(\zeta)\, d\Omega(\zeta; z, \infty^{(-)}) + c_n. \tag{19}$$

Note that the integral here is not singular anymore, therefore $wR\varphi$ has a continuous limit on Δ and (19) is valid also for $z \in \Delta$. Thus, (19) implies

$$M_n = |(wR\varphi)(z_0)| = \left|\frac{1}{2\pi} \int_{\Delta'} (wR\varphi)(\zeta) \left(\frac{\psi}{\varrho\phi}\right)(\zeta)\, d\Omega(\zeta; z_0, \infty^{(-)}) + c_n\right|$$

for some $z_0 \in \Delta$. Taking into account the expression for $\frac{\psi}{\varphi}$ (see (7) and (11)) and the fact that $d\Omega$ has no singularities in $\mathcal{R}^{(-)} \backslash (\Delta \cup \{\infty^{(-)}\})$, we find

$$M_n \leqslant M_n C(1-\varepsilon)^n + c_n \Rightarrow M_n \leqslant \frac{c_n}{1 - C(1-\varepsilon)^n}. \tag{20}$$

Now we can get *the asymptotics of Q on compact sets* $K \subset \overline{\mathbb{C}} \backslash [-1, 1]$. Fix K and choose ε such that $K \subset \overline{\mathbb{C}} \backslash \mathcal{A}$. Considering (18) for $z \in \mathcal{R}^{(-)} \backslash \mathcal{A}$, we have

$$(Q\psi)(z) - c_n = -\frac{1}{2\pi i} \int_{\Delta'} (h\varphi R)(\zeta) \left(\frac{\psi}{\varrho\varphi}\right)(\zeta)\, d\Omega(\zeta; z, \infty^{(-)}).$$

Therefore,

$$||Q\psi - c_n||_{C(\mathcal{K})} \leqslant \widetilde{C} M_n (1-\varepsilon)^n \leqslant c_n \frac{\widetilde{C}}{1 - C(1-\varepsilon)^n} (1-\varepsilon)^n.$$

Dividing by c_n, we obtain the desired asymptotics:

$$\frac{(Q\psi)(z)}{c_n} = 1 + O(q_K^n), \quad q_K < 1,$$

or

$$Q(z) = c_n \varphi(z)(1 + O(q_K^n)).$$

3.4. Asymptotics on $[-1, 1]$: statement of the theorem

As we already mentioned, the integral equation (19) remains valid for $z \in \varDelta$. Thus

$$f_+(z)|_{z\in\varDelta} = -\frac{1}{2\pi i} \int_{\varDelta'} (wR\varphi)(\zeta) \left(\frac{\psi}{\varrho\varphi}\right)(\zeta)\, d\Omega(\zeta; z, \infty^{(-)}) + c_n.$$

Therefore, for $z \in \varDelta$

$$|f_+(z) - c_n| = \left| \frac{1}{2\pi} \int_{\varDelta'} (wR\varphi)(\zeta) \left(\frac{\psi}{\varrho\varphi}\right) d\Omega(\zeta; z, \infty^{(-)}) \right|$$
$$\leqslant M_n C(1-\varepsilon)^n \leqslant c_n \frac{C(1-\varepsilon)^n}{1 - C(1-\varepsilon)^n}.$$

So, we have obtained uniform asymptotics for the remainder $R(z)$ on the interval $[-1, 1]$:

$$\left.\frac{f_+(x)}{c_n}\right|_{x\in\varDelta} = \left.\frac{(wR\varphi)_\pm(x)}{c_n}\right|_{x\in[-1,1]} = 1 + O(q^n),$$

where $0 < q < 1$ and q depends on the size of the domain of analyticity of $\varrho(z)$, (i.e., on δ).

Now, for the polynomials Q we have from the boundary value Padé problem (15)

$$Q = \left(\frac{wR}{\varrho}\right)_+ + \left(\frac{wR}{\varrho}\right)_- \quad \text{on } [-1, 1]$$

and because of

$$\frac{(wR)_\pm}{\varrho} = \frac{c_n}{\varphi_\pm \varrho}(1 + O(q^n)) = \frac{c_n}{\psi_\mp}(1 + O(q^n)) = c_n \varphi_\mp (1 + O(q^n)),$$

we finally obtain (taking into account the boundedness of φ on $[-1, 1]$)

$$\frac{Q}{c_n} = \varphi_+ + \varphi_- + O(q^n) \quad \text{on } [-1, 1].$$

Now we can make the following remark about the normality of Padé approximants. We can omit our assumption for ϱ that $\deg Q = n$, for n large enough. Indeed, if $\deg Q < n$, then everything above remains valid apart from condition 3 in the Riemann problem (16), which becomes

$$3' \ \mathcal{F}(\infty^{(-)}) = c_n = 0,$$

but this leads to a contradiction with (20) for large n.

Thus we have proved the following theorem:

Theorem 1. *Let ϱ be a non-vanishing complex valued function on $[-1,1]$, which is analytic in some δ neighborhood of $[-1,1]$. Then for the Markov function*

$$\widehat{\varrho}(z) := \frac{1}{2\pi}\int_{-1}^{1} \frac{\varrho(\zeta)}{z-\zeta}\frac{d\zeta}{\sqrt{1-\zeta^2}}$$

and sufficiently large n, there exists a unique polynomial Q, with $Q(z) = z^n + \cdots$, such that

$$Q\widehat{\varrho} + P = R, \quad \deg P = n-1, \quad R(z)|_{z\to\infty} = O\left(\frac{1}{z^{n+1}}\right).$$

The polynomial Q and the remainder function R have the following asymptotic formulas:

$$1)\ \frac{Q(z)}{c_n \varphi(z)} = 1 + O(q^n_{K,\varrho}), \quad z \in K \subset \overline{\mathbb{C}} \setminus [-1,1],$$

$$\frac{Q(x)}{c_n} = \varphi_+(x) + \varphi_-(x) + O(q^n_\varrho), \quad x \in [-1,1],$$

$$2)\ \frac{R_\pm(x)}{c_n} = \frac{1}{(\sqrt{x^2-1}\varphi(x))_\pm} + O(q^n_\varrho), \quad x \in [-1,1],$$

where

$$\varphi(z) = (z+\sqrt{z^2-1})^n \exp\left\{-\frac{1}{2\pi}\sqrt{z^2-1}\int_{-1}^{1}\frac{\ln\varrho(x)}{z-x}\frac{dx}{\sqrt{1-x^2}}\right\}$$

and

$$c_n = \left(\frac{1}{2}\right)^n \exp\left\{\frac{1}{2\pi}\int_{-1}^{1}\frac{\ln\varrho(x)}{\sqrt{1-x^2}}dx\right\}.$$

The constants $q_{K,\varrho}$, q_ϱ are such that $0 < q_{K,\varrho}, q_\varrho < 1$ and depend on the compact set K and on the size of the domain of analyticity of ϱ (i.e., on δ).

4. Strong asymptotics for polynomials with respect to a varying weight

4.1. Conditions on the varying weight. Equilibrium in external field. Statement of the problem

Here we assume the weight generating a Markov function in (12) has a dependence on n of the form

$$\rho_n := e^{-2nq_n}\rho, \quad \rho \in H(\delta), \quad \rho \neq 0 \text{ on } [-1,1] \subset \delta$$

and for q_n we assume

$$||q_n - q||_{C(\delta)} = O\left(\frac{1}{n}\right), \quad q_n \in H(\delta), \quad \Im q = 0 \text{ on } [-1,1].$$

These settings for ρ_n can be rewritten as

$$\rho_n = e^{-2nq}\tilde{\rho}_n, \quad \rho_n \in H(\delta), \quad \rho_n \neq 0 \text{ in } \delta$$

and

$$||\tilde{\rho}_n - \rho||_{C(\delta)} = O(1).$$

We note that for further consideration it is enough to assume that $\{\tilde{\rho}_n\}$ is a compact family in $H(\delta)$.

Next we have to put extra conditions on q. For this we consider the equilibrium problem in the presence of the external field q for the logarithmic potential

$$V_\mu(z) = -\int_{-1}^{1} \ln|x-z|\, d\mu(x)$$

of positive probability measures $\mu \in \mathcal{M}^+([-\infty,\infty])$, supported on $[-1,1]$. It is well-known (see [9,15]) that there exists a unique measure λ (called *equilibrium measure in external field*) such that

$$V_\lambda + q = \begin{cases} \gamma & \text{on supp } \lambda, \\ \geqslant \gamma & \text{on } [-1,1]. \end{cases} \tag{21}$$

In what follows we assume

$$q \text{ is such that supp } \lambda = [-1,1]. \tag{22}$$

Remark. A sufficient condition for (22) is convexity of q on $[-1,1]$. It is not so difficult to see (see, for example [9]), that (22) and $q \in H(\delta)$ imply absolute continuity of λ and

$$\lambda' \in \mathcal{A}(\delta \backslash (\{-1\} \cup \{+1\})),$$

i.e., λ' has analytic continuation in the punctured (at ± 1) domain δ. In what follows we assume more, namely that

$$q \text{ is such that } \lambda'(z) = \frac{m(z)}{\sqrt{z^2-1}}, \quad m \in H(\delta),\ m(\pm 1) \neq 0. \tag{23}$$

Remark. If the equilibrium measure for the problem on $\mathcal{M}^+(I)$, for some I : $[-1,1] \subset I \subset \delta$ has its support equal to I, then condition (23) is fullfiled. A proof of these remarks can be found in [9]. We also mention that if q is a convex analytic function on $\delta \cap R$, then both our extra conditions (22) and (23) hold true (this follows from the previous two remarks).

Now we state a problem about polynomials orthogonal with respect to the varying weight. These polynomials have a variety of very important applications(see, e.g., [19]). Suppose the family of Markov functions

$$\hat{\rho}_n(z) = \frac{1}{2\pi} \int_{-1}^{1} \frac{\rho_n(x)\,dx}{(z-x)\sqrt{1-x^2}}$$

is generated by the family of weight functions of the form

$$\rho_n = e^{-2nq}\tilde{\rho}_n, \quad \rho_n \in H(\delta), \quad \rho_n \neq 0 \text{ in } \delta, \quad \Im q = 0 \text{ on } [-1,1],$$

where q is such that the equilibrium measure λ of problem (21) satisfies conditions (22), (23) and $\{\tilde{\rho}_n\}$ is a compact family in $H(\delta)$. For each n we consider Padé approximants of index n to the function $\hat{\rho}_n$

$$Q_n(z) = z^n + \cdots, \quad Q_n\hat{\rho}_n + P_n = R_n = O\left(\frac{1}{z^{n+1}}\right).$$

The polynomial denominators Q_n satisfy the following orthogonality relations:

$$\int_{-1}^{1} Q_n(x)\rho_n(x)x^{\nu}\,dx = 0, \quad \nu = 0, \ldots, n-1.$$

The investigation of the strong asymptotics for these polynomials Q_n and for the remainder function R_n goes along the same lines as we did in the previous chapter for non-varying weight. Only two new ingredients will be introduced. In the next section, using the equilibrium problem (21) we present another expression for the solution of the auxiliary BVP (1) for a varying weight. Then in Section 4.3 we do a more delicate analytic continuation of the jump function for the remainder function's Riemann problem (16) on the second sheet $\mathcal{R}^{(-)}$ of the Riemann surface.

4.2. *Another representation for the solution of the auxiliary BVP for varying weight ρ_n*

Here we consider the solution of BVP problem (1) for the varying weight ρ_n. As we know (see (11) and (10)) the solution is

$$\varphi(z) = \Phi^n(z)\exp\left\{-\frac{n}{2\pi i}\int_{-1}^{1} \frac{w(z)}{w_+(x)}\frac{2q(x)}{z-x}\,dx + \frac{1}{2\pi i}\int_{-1}^{1} \frac{w(z)}{w_+(x)}\frac{\ln\tilde{\rho}_n(x)}{z-x}\,dx\right\},$$

$$\psi(z) = \frac{1}{\varphi(z)}.$$

We denote by $\mathcal{G} = \ln \Phi$ the complex Green function, H is the first integral in the exponential above and $F^{(\tilde{\rho}_n)}$ stands for the last multiplier of φ:

$$\varphi =: \exp\{n(\mathcal{G} - H)\}F^{(\tilde{\rho}_n)} \tag{24}$$

We will also use the notation

$$F^{(\rho_q)} := e^{-H} \quad \varphi^{(q)} := \exp\{n(\mathcal{G} - H)\} \quad \text{and} \quad \psi^{(q)} = \frac{1}{\varphi^{(q)}}.$$

From (3) and (8) it follows that on $[-1, 1]$

$$\begin{aligned} &\varphi_+^{(q)}\varphi_-^{(q)}e^{-2nq} = 1, \\ &F_+^{(\rho_q)}F_-^{(\rho_q)}e^{-2q} = 1. \end{aligned} \tag{25}$$

The last relation gives us

$$H_+ + H_- + 2q = 0 \quad \text{on } [-1, 1],$$

which, taking into account the *symmetry with respect to* $\mathbb{R}$ leads to

$$\Re H + q = 0 \quad \text{on } [-1, 1].$$

Thus the harmonic function

$$h := \Re H \quad \text{in } \overline{\mathbb{C}}$$

gives a solution of the Dirichlet problem

$$h : \begin{cases} h \in \operatorname{Harm}(\overline{\mathbb{C}} \backslash [-1, 1]) \\ h = -q \quad \text{on } [-1, 1]. \end{cases} \tag{26}$$

Now we turn to the equilibrium problem (21)

$$V_\lambda(x) + q(x) = \gamma, \quad x \in [-1, 1].$$

Using a Green function $g(z)$ of the domain $\overline{\mathbb{C}} \backslash [-1, 1]$:

$$g : \begin{cases} g \in \operatorname{Harm}(\mathbb{C} \backslash [-1, 1]), \\ g = \ln|z| + C + \cdots \quad \text{near } z = \infty, \\ g = 0 \quad \text{on } [-1, 1], \end{cases}$$

and the solution of the Dirichlet problem (26), we rewrite the equilibrium relation as

$$V_\lambda(x) - \gamma = h(x) - g(x), \quad x \in [-1, 1].$$

In fact the above relation holds not only on $[-1, 1]$, but on the whole $\overline{\mathbb{C}}$. Indeed, the difference between the right and the left-hand sides of the relation is a harmonic function on $\overline{\mathbb{C}} \backslash [-1, 1]$ (the singularities at ∞ are canceled) and its boundary values on $[-1, 1]$ are equal to zero, therefore by the maximum principle for harmonic functions, we have that the difference is zero on $\overline{\mathbb{C}}$. Thus

$$V_\lambda(z) - \gamma = h(z) - g(z), \quad z \in \overline{\mathbb{C}} \tag{27}$$

and adding complex conjugate functions to (27), we obtain an identity

$$\mathcal{V}_\lambda(z) - \gamma = H(z) - \mathcal{G}(z),$$

where $\mathcal{V}_\lambda$ stands for the complex potential of the measure λ.

Thus, substituting the obtained identity in (24), we have a new representation for the solution of BVP problem (1) for $\rho := \rho_n$

$$\begin{cases} \varphi = \varphi^{(q)} F^{(\tilde\rho_n)} = \exp\{n(\gamma - \mathcal{V}_\lambda)\} F^{(\tilde\rho_n)}, \\ \psi = \dfrac{1}{\varphi}. \end{cases} \tag{28}$$

4.3. Integral equation for the remainder function for the case of a varying weight

As in the case of a non-varying weight we can use the function satisfying the auxiliary BVP (1) with ρ_n to arrive at the integral equation (see (17))

$$f_n(z) = -\frac{1}{2\pi i}\int_\Delta \left(\frac{\psi w R_n}{\rho_n}\right)(\xi)\, d\Omega(\xi; z, \infty^{(-)}) + C_n, \quad z \in \mathcal{R},$$

where

$$f_n = \begin{cases} wR_n\varphi & \text{on } \mathcal{R}^{(+)}, \\ Q\psi & \text{on } \mathcal{R}^{(-)} \end{cases} \tag{29}$$

and as contour Δ we choose a cut along the upper and lower sides of the interval $[-1,1] \subset \mathcal{R}^{(-)}$ with negative orientation with respect to the $\mathcal{R}^{(-)}$ direction.

As before we would like to deform the contour Δ to the inside of the second sheet $\mathcal{R}^{(-)}$ using analytic continuation of the jump function. The aim of this section to show that, under our conditions on the varying weight, the analytic continuation of the jump $\left(\frac{\psi w R_n}{\rho_n}\right)$ on the contour $\Delta' \subset \mathcal{R}^{(-)} \setminus [-1,1]$ decreases exponentially when $n \to \infty$. We consider the analytic continuation of

$$\frac{\psi w R_n}{\rho_n} = \left(\frac{wR_n}{\psi}\right)\left(\frac{\psi^2}{\rho_n}\right) \quad \text{from } [-1,1]_\pm \text{ to } \overline{\mathbb{C}}$$

and will pay special attention to the continuation of the second factor. Taking into account the multiplicative dependence of the solution of BVP (10) on weight functions, and using the notation of the previous subsection, we have

$$\frac{\psi^2}{\rho_n} = \frac{(F^{(\tilde\rho_n)})^{-2}}{\tilde\rho_n}\frac{(\psi^{(q)})^2}{e^{-2nq}} \quad \text{on } [-1,1]_\pm.$$

We consider the continuation of the second factor. We have (see (25))

$$\frac{\left(\psi_\pm^{(q)}\right)^2}{e^{-2nq}} = \frac{\psi_\pm^{(q)}\varphi_\mp^{(q)}}{\varphi_\pm^{(q)}\varphi_\mp^{(q)} e^{-2nq}} = \psi_\pm^{(q)}\varphi_\mp^{(q)} = \frac{\varphi_\mp^{(q)}}{\varphi_\pm^{(q)}} \quad \text{on } [-1,1]$$

and applying the new representation for the solution of BVP (28) for a varying weight, we find

$$\frac{\left(\psi_{\pm}^{(q)}\right)^2}{e^{-2nq}} = \exp\{-n(\mathcal{V}_{\lambda\mp} - \mathcal{V}_{\lambda\pm})\} \quad \text{on } [-1,1]$$

and because of the *symmetry with respect to* $\mathbb{R}$, we have

$$\frac{\left(\psi_{\pm}^{(q)}\right)^2}{e^{-2nq}} = \exp\{2ni\ \Im\mathcal{V}_{\lambda\pm}\}, \quad \text{on } [-1,1].$$

Now we shall prove that under condition (23) a function

$$\ell = \frac{1}{\pi}\Im\mathcal{V}_{\lambda} \quad \text{on } [-1,1]_{\pm} \tag{30}$$

has a holomorphic continuation to $\{\delta\backslash[-1,1]\}$ and for some $\tilde{\delta}\subset\delta$

$$\Im\ell(z)>0, \quad z\in\tilde{\delta}\backslash[-1,1]. \tag{31}$$

We do this in three steps.

1. First we present an expression for the function $\ell(x)$, $x\in[-1,1]$. We have

$$\mathcal{V}_{\pm}(\lambda) = -\int_{-1}^{1} \ln(z-t)\lambda'(t)\,dt = -\int_{-1}^{1} \ln|z-t|\lambda'(t)\,dt$$
$$-\,i\int_{-1}^{1} \arg(z-t)\lambda'(t)\,dt.$$

   Fixing a branch $\arg(\xi)=0$, $\xi>0$ we analytically continue the above formula from some point $z\in[1,\infty]$ to some point $x\in[-1,1]$ along some path belonging to the upper half plane and along some path from the lower half plane. As a result we will have

$$\mathcal{V}_{\pm}(x) = V(x)\mp i\pi\int_{x}^{1} \lambda'(t)\,dt, \quad x\in[-1,1].$$

   Thus

$$\ell_{\pm}(x) = \mp\int_{x}^{1} \lambda'(t)\,dt.$$

2. Then we see that the function

$$\ell(z) = -\int_{z}^{1} \lambda'(\xi)\,d\xi \tag{32}$$

   gives a holomorphic continuation of $\ell_{\pm}(x)$ in the upper and lower neighborhood of $[-1,1]$. We denote these lense-shaped simply connected domains as $\tilde{\delta}_{+}$ and $\tilde{\delta}_{-}$.

There the function $\ell(z)$ is a primitive of the holomorphic branch of the analytic function $\lambda'(z)$

$$\ell_{\pm}(z) := -\int_z^1 \lambda'(\xi)\, d\xi, \quad z \in \tilde{\delta}_{\pm}.$$

Using a local representation of $\ell(z)$ in the neighborhoods O_{+1} and O_{-1} of the end points of $[-1,1]$ (which follows from (32) and (23))

$$\ell(z) = \begin{cases} m_{+1}(z)\sqrt{1-z}, & m_{+1} \in H(O_{+1}), \quad \Im m_{+1} = 0 \text{ on } \mathbb{R} \cap O_{+1},\ z \in O_{+1}, \\ m_{-1}(z)\sqrt{z-1}, & m_{-1} \in H(O_{-1}), \quad \Im m_{-1} = 0 \text{ on } \mathbb{R} \cap O_{-1},\ z \in O_{-1}. \end{cases} \tag{33}$$

we see that analytic continuation $\ell_{+}(z)$ from $\tilde{\delta}_{+}$ to some point $x \in \mathbb{R}$ such that $x \in ((Q_{+1} \cup O_{-1}) \backslash [-1,1]) \cap \mathbb{R})$ coincides with the analytic continuation of $\ell_{-}(z)$ from $\tilde{\delta}_{-}$ to the same point x. Thus the function (32) is holomorphic in a neighborhood of $[-1,1]$

$$\ell \in H(\tilde{\delta} \backslash [-1,1]), \quad \tilde{\delta} := \tilde{\delta}_{+} \cup \tilde{\delta}_{-} \cup O_{+1} \cup O_{-1}$$

and satisfies the boundary condition (30) on $[-1,1]$.

3. It remains to check (31). In the domains $\tilde{\delta}_{+}$ and $\tilde{\delta}_{-}$ inequality (31) is true because of the Cauchy–Riemann equations

$$\frac{\partial \Im \ell}{\partial y} = \frac{\partial \Re \ell}{\partial x} = \lambda'(x) > 0.$$

To check (31) in the domain $(O_{+1} \cup O_{-1}) \backslash [-1,1]$, we use the local representation (33) of $\ell(z)$ there. Take for example O_{+1}, we can choose O_{+1} small enough such that the argument of m_{+1} (which is zero on $\mathbb{R} \cap O_{+1}$) does not make a substantial contribution to the argument of $\ell(z)$, so

$$\arg \ell_{-}(x) = 0 < \arg \ell(z) < \pi = \arg \ell_{+}(x), \quad z \in O_{+} \backslash [-1,1], \ \ x \in [-1,+1].$$

Thus

$$\Im \ell > 0 \ \text{in} \ (O_{+} \cup O_{-1}) \backslash \Delta$$

and we have (31) for ℓ given by (32).

Summarizing our transformations of the jump function in (29), we have

$$j = \frac{w\psi R_n}{\rho_n} = \frac{wR_n}{\psi} J, \quad J := \frac{\left(F^{(\tilde{\rho}_n)}\right)^{-2}}{\tilde{\rho}_n} e^{2ni\ell} \ \text{on} \ \Delta = [-1,1]_{\pm} \subset \mathcal{R}.$$

So now we can lift the domain $\tilde{\delta}$ on the sheet $\mathcal{R}^{(-)}$, we denote it by

$$\tilde{\mathcal{D}}^{(-)} \subset \mathcal{R}^{(-)}, \quad \pi(\tilde{\mathcal{D}}^{(-)}) = \tilde{\delta}$$

and using analytic continuation of j to $\tilde{\mathcal{D}}^{(-)}$ it is possible to deform contour Δ in (29) to some contour $\Delta' \subset \tilde{\mathcal{D}}^{(-)}$:

$$f_n(z) = -\frac{1}{2\pi i}\int_{\Delta'}\left(\frac{wR_n}{\psi}(\xi)J(\xi)\,d\Omega(\xi; z, \infty^{(-)})\right) + C_n, \quad z \in \mathcal{R}\backslash\mathcal{A}, \tag{34}$$

where $\mathcal{A} \subset \mathcal{R}^{(-)}$ is a ring domain bounded by $\Delta \cup \Delta'$.

Finally, taking into account compactness of $\tilde{\rho}_n$ in $H(\tilde{\delta})$ and (31) we have

$$||J||_{C(\Delta')} \leqslant C\kappa^n, \quad \kappa < 1, \tag{35}$$

where the constants C and κ depend on analytic properties of the varying weight ρ_n.

4.4. Statement of the theorem (for varying weight)

Thus using estimation (35) in the integral equation (34) and repeating all arguments we used to prove the corresponding theorem for the non-varying weight, we arrive at the following theorem

Theorem 2. *Let $\{\rho_n\}$ be a family of holomorphic functions in the domain $\delta \supset [-1, 1]$*

$$\rho_n = e^{-2nq}\tilde{\rho}_n, \quad \rho_n \in H(\delta), \quad \rho_n \neq 0 \ \text{in} \ \delta \supset [-1, 1],$$

where $\{\tilde{\rho}_n\}$ is a compact family in $H(\delta)$, and q is real valued on $[-1, 1]$,

$$\Im q = 0 \quad \text{on} \ [-1, 1].$$

Furthermore, ρ_n is such that the equilibrium measure λ in the external field q

$$V_\lambda + q = \begin{cases} \gamma & \text{on supp}\,\lambda, \\ \geqslant \gamma & \text{on } [-1, 1] \end{cases}$$

is absolutely continuous on supp $\lambda = [-1, 1]$ *and its derivative has the form*

$$\lambda'(x) = \frac{m(x)}{\sqrt{1 - x^2}}, \quad m \in H(\delta), \ m(\pm 1) \neq 0.$$

Then for sufficiently large n, there exists a unique polynomial

$$Q_n(z) = z^n + \cdots,$$

which is orthogonal with respect to the weight function ρ_n

$$\int_{-1}^{1} Q_n(x)x^\nu \frac{\rho_n(x)\,dx}{\sqrt{1 - x^2}} = 0, \quad \nu = 0, \ldots, n - 1$$

and for the polynomials Q_n and the functions of the second kind

$$R(z) = \int_{-1}^{1} \frac{Q_n(x)}{z - x}\frac{\rho_n(x)\,dx}{\sqrt{1 - x^2}}$$

the following asymptotics formulas hold:

$$\frac{Q_n(z)}{c_n\varphi(z)} = 1 + O(\kappa_K^n), \quad z \in K \subset \overline{\mathbb{C}}\backslash[-1, 1],$$

$$\frac{Q_n(x)}{c_n} = \varphi_+(x) + \varphi_-(x) + O(\kappa_1^n), \quad x \in [-1, 1]$$

$$\frac{R_\pm(x)}{c_n} = \frac{1}{(\sqrt{x^2-1}\varphi(x))_\pm} + O(\kappa_2^n), \quad x \in [-1, 1],$$

where the constants $\kappa \in (0, 1)$, φ, *and* c_n *are*

$$\varphi(z) = e^{n(\gamma - \mathcal{V}_\lambda(z))} \exp\left\{ -\frac{1}{2\pi}\sqrt{z^2-1} \int_{-1}^{1} \frac{\ln \tilde{\rho}_n(x)\, dx}{(z-x)\sqrt{1-x^2}} \right\}$$

$$c_n = e^{-n\gamma} \exp\left\{ \int_{-1}^{1} \frac{\ln \tilde{\rho}_n(x)}{\sqrt{1-x^2}}\, dx \right\},$$

and

$$\gamma = \ln 2 + \frac{1}{\pi} \int_{-1}^{1} \frac{q(x)}{\sqrt{1-x^2}}\, dx.$$

5. A matrix Riemann–Hilbert problem approach

We start here with the matrix Riemann–Hilbert problem formulation of the orthogonality relations. Then we recall the auxiliary boundary value problem, which we studied in Section 2 and which will be used here for the normalization of the matrix Riemann–Hilbert problem. Next, according to Deift [3], we proceed with the transformation of the original matrix Riemann–Hilbert problem to a problem with a jump matrix function which tends to the identity matrix as n (the degree of the polynomials) tends to infinity. Finally, to make the presentation self contained, we give a proof of the lemma stating that, if the jump matrix for the matrix valued homogeneous Riemann–Hilbert problem tends to the identity matrix, then the solution also tends to the identity matrix. At this point, we again take a shortcut by assuming that the jump matrix is analytic, it is possible for us to give a trivial proof, just based on the Cauchy theorem, without applying the harmonic analysis which was used in its original version in [3].

5.1. Orthogonal polynomials and a matrix Riemann–Hilbert problem

Let

$$Q_n(x) := x^n + \cdots$$

be a monic orthogonal polynomial

$$\int_{-1}^{1} Q_n(x) x^k h(x)\, dx = 0, \quad k = 0, \ldots, n-1, \tag{36}$$

with respect to a weight function h, which we assume to have the form

$$h(x) := \frac{\varrho(x)}{\sqrt{1-x^2}}, \quad x \in [-1,1], \tag{37}$$

where ϱ is a complex valued function, non-vanishing on $[-1,1]$, which is holomorphic in some domain δ containing the interval $[-1,1]$:

$$\varrho \neq 0 \text{ on } [-1,1], \quad \varrho \in H(\delta), \ [-1,1] \subset \delta. \tag{38}$$

Let R_n be the function of the second kind associated with Q_n:

$$R_n(z) = \frac{1}{2\pi i} \int_{-1}^{1} \frac{Q_n(x)h(x)\,dx}{x-z}, \quad R_n \in H(\overline{\mathbb{C}} \setminus [-1,1]). \tag{39}$$

It is easy to verify that the orthogonality relations for Q_n are equivalent to the fact that

$$R_n(z) = O\left(\frac{1}{z^{n+1}}\right), \quad z \to \infty. \tag{40}$$

As before (see Section 3), our starting point is the Sokhotsky–Plemelj formula for the boundary values of a Cauchy type integral, which we apply to the function of the second kind (39)

$$R_{n+} - R_{n-} = hQ_n \quad \text{on } [-1,1]. \tag{41}$$

Here, as usually, $(+)$ denotes the boundary values of the function from the upper side of $[-1,1]$, and $(-)$ from the lower side. Choosing m as a normalization constant so that

$$mR_{n-1} = \frac{1}{z^n} + \cdots, \quad z \to \infty,$$

i.e.,

$$m = -\frac{2\pi i}{\int_{-1}^{1} Q_{n-1}^2(x)h(x)\,dx}$$

and applying (41) to R_{n-1}, then for the matrix valued analytic function

$$Y = \begin{pmatrix} Q_n & R_n \\ mQ_{n-1} & mR_{n-1} \end{pmatrix}, \tag{42}$$

we obtain the following Riemann–Hilbert problem:

$$Y := \begin{cases} Y \in H(\mathbb{C} \setminus [-1,1]), \\ Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \operatorname{diag}(z^n, z^{-n}) \quad \text{as } z \mapsto \infty, \\ Y_+ = Y_- \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \quad \text{on } [-1,1]. \end{cases} \tag{43}$$

Our goal is to show that solution (42) of (43), which depends on n, tends as $n \to \infty$ to a solution of some boundary value problem which does not depend on n.

First we mention that our restriction for the weight function to be of the form (37) and (38) allows us to make an analytic continuation of $h(x)$ in the domain $\delta\backslash[-1,1]$

$$h(z) := \frac{\varrho(z)}{i(z^2-1)^{\frac{1}{2}}} \in H(\delta\backslash[-1,1]), \tag{44}$$

where the branch for $(z^2-1)^{\frac{1}{2}}$ is chosen such that

$$(z^2-1)^{\frac{1}{2}} > 0 \quad \text{for } z > 1.$$

It gives for the limiting values of $h(z)$

$$h_\pm(x) = \mp \frac{\varrho(x)}{\sqrt{1-x^2}} \quad \text{on } [-1,1]$$

and if we denote

$$W(z) = \begin{pmatrix} 1 & h(z) \\ 0 & 1 \end{pmatrix},$$

then we have

$$W_- = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}, \quad W_+ = \begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix} = W_-^{-1}. \tag{45}$$

The latter remarkable relation allows us to present the jump condition in (43) as

$$Y_+ = Y_- W_- \quad \text{or} \quad Y_- = Y_+ W_+ \quad \text{on } [-1,1].$$

5.2. *Auxiliary boundary value problem recall*

Before stating an asymptotic result for the solution of the matrix Riemann–Hilbert problem, we recall the auxiliary BVP which we studied in Section 2. In terms of the solution of this BVP we will write an answer (i.e., final asymptotic formulas) and the properties of the solution will be used in the proof. The problem consist on finding φ such that

$$\varphi : \begin{cases} \varphi \in H(\mathbb{C}\backslash[-1,1]), \quad \exists \varphi_+, \varphi_- \in C([-1,1]), \\ \varphi(z) = O(z^n), \quad z \to \infty, \quad \left.\dfrac{\varphi(z)}{z^n}\right|_\infty > 0, \\ \varphi_+ \varphi_- \varrho = 1 \quad \text{on } [-1,1]. \end{cases} \tag{46}$$

The unique solution of problem (46) is (see (10) and (11))

$$\varphi(z) = (z + \sqrt{z^2-1})^n \exp\left\{ -\frac{1}{2\pi} \sqrt{z^2-1} \int_{-1}^{1} \frac{\ln \varrho(x)}{z-x} \frac{dx}{\sqrt{1-x^2}} \right\} \tag{47}$$

and the normalization constant for $\varphi|_\infty = \frac{z^n}{c} + \cdots$ is

$$c = \frac{1}{2^n} \exp\left\{ \frac{1}{2\pi} \int_{-1}^{1} \frac{\ln \varrho(x)}{\sqrt{z^2-1}} dx \right\}. \tag{48}$$

We also note that the scalar problem (46) admits several equivalent reformulations, i.e., the vector version

$$\begin{cases} (\varphi,\psi)\in H(\mathbb{C}\setminus[-1,1]), \\ (\varphi,\psi)(z) = \left(\frac{z^n}{c}+\cdots,\frac{c}{z^n}+\cdots\right) \quad \text{for some } c>0,\ n\to\infty, \\ (\varphi,\psi)_+ = (\varphi,\psi)_-\begin{pmatrix} 0 & \varrho \\ \varrho^{-1} & 0 \end{pmatrix} \quad \text{on } [-1,1] \end{cases}$$

and the matrix version

$$\begin{cases} \Psi = \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}, \quad \varphi,\psi \text{ as above}, \\ \Psi_+ = \Psi_-^T\begin{pmatrix} \varrho & 0 \\ 0 & \varrho^{-1} \end{pmatrix}. \end{cases}$$

We are not going to use these versions here and mention them just for completeness.

5.3. Statement of the asymptotic result

Here we prove the following

Theorem 3. *Suppose that the weight function* (37) *satisfies conditions* (38). *Then for the matrix* Y (*see* (42)) *of the orthogonal polynomials* (36) *and for the functions of the second kind* (39), *the following asymptotic formula holds*:

$$CYS(z) = \left(I+\frac{O(q^n)}{z}\right)X, \quad n\to\infty,\ 0<q<1, \tag{49}$$

uniformly outside any Jordan contour $\Gamma\subset\delta$ *around* $[-1,1]$, *and*

$$CYS(z) = (I+O(q^n))XD^{-1}, \tag{50}$$

uniformly inside the domain bounded by Γ, *where*

$$C = \operatorname{diag}(c^{-1},c), \quad S = \operatorname{diag}(\varphi^{-1},\varphi) \tag{51}$$

are given by (46) *and* (48), *and*

$$D = \begin{pmatrix} 1 & 0 \\ (h\varphi^2)^{-1} & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 1 & \frac{i}{(z^2-1)^{\frac{1}{2}}} \\ \frac{\left(z-(z^2-1)^{\frac{1}{2}}\right)}{2i} & \frac{z+\left(z^2-1\right)^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}} \end{pmatrix}.$$

The matrix asymptotic formulas (49) and (50) lead to the following asymptotic formulas for the polynomials Q_n and for the functions of the second kind R_n

$$1.\ \frac{Q_n(z)}{c\varphi(z)} = 1 + O(q^n), \quad z \in K \subset \overline{\mathbb{C}} \setminus [-1,1],$$
$$\frac{Q_n(x)}{c} = \varphi_+(x) + \varphi_-(x) + O(q^n), \quad x \in [-1,1],$$
$$2.\ \frac{R_{n\pm}(x)}{c} = \frac{1}{(\sqrt{x^2-1}\varphi(x))_\pm} + O(q^n), \quad x \in [-1,1].$$

5.4.. *Proof of the theorem*

1. First we normalize the Riemann–Hilbert problem (43), forcing the solution to be holomorphic at ∞. We define (see (42) and (51))

$$Z := CYS = \begin{pmatrix} \dfrac{Q_n}{c\varphi} & \dfrac{R_n\varphi}{c} \\ \dfrac{cmQ_{n-1}}{\varphi} & cm\varphi R_{n-1} \end{pmatrix}. \tag{52}$$

Now the matrix-valued function Z is holomorphic in $\overline{\mathbb{C}} \setminus [-1,1]$ and it satisfies the following Riemann–Hilbert problem:

$$\begin{cases} Z \in H(\overline{\mathbb{C}} \setminus [-1,1]), \\ Z_+ = Z_- J \quad \text{on } [-1,1], \\ Z(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty, \end{cases} \tag{53}$$

where the jump matrix J, because of

$$(C^{-1}ZS^{-1})_+ = (C^{-1}ZS^{-1})_- W_- \quad \text{on } [-1,1]$$

is given by

$$J = S_-^{-1} W_- S_+ = \begin{pmatrix} \dfrac{\varphi_-}{\varphi_+} & \varphi_+\varphi_- h \\ 0 & \dfrac{\varphi_+}{\varphi_-} \end{pmatrix} \quad \text{on } [-1,1].$$

Our choice of the solution of the auxiliary BVP (46) for the normalization of problem (43) gives

$$J = \begin{pmatrix} \frac{\varphi_-}{\varphi_+} & \frac{1}{\sqrt{1-x^2}} \\ 0 & \frac{\varphi_+}{\varphi_-} \end{pmatrix} \quad \text{on } [-1,1].$$

This 'lucky' expression for the jump matrix of problem (53) allows us to decompose it as

$$\begin{aligned} J &= \begin{pmatrix} 1 & 0 \\ \frac{\varphi_+}{\varphi_-}\sqrt{1-x^2} & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{1-x^2}} \\ -\sqrt{1-x^2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\varphi_-}{\varphi_+}\sqrt{1-x^2} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \frac{1}{(h\varphi^2)_-} & 1 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\sqrt{1-x^2}} \\ -\sqrt{1-x^2} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{-1}{(h\varphi^2)_+} & 1 \end{pmatrix}. \end{aligned} \tag{54}$$

We see that the first matrix here admits an holomorphic continuation in the lower half of the complex plane and the last matrix does the same in the upper half plane. After the analytic continuation, and because of the exponential decrease of φ^{-1} outside $[-1,1]$ as $n \to \infty$, we see that J will be close to a very 'friendly' central matrix. This is the essence of the method!

2. Now, in order to develop this idea about the analytic continuation of the parts of the jump, we transform the Riemann–Hilbert problem (53) into the following problem. We denote

$$D := \begin{pmatrix} 1 & 0 \\ (h\varphi^2)^{-1} & 1 \end{pmatrix} \in H(\delta \backslash [-1,1]) \tag{55}$$

and let Γ be a contour in δ such that $\Delta := [-1,1]$ is in the domain $\mathrm{Int}(\Gamma)$ which is bounded by Γ and $\infty \notin \mathrm{Int}(\Gamma)$:

$$\Gamma \subset \delta, \qquad \Delta \in \mathrm{Int}(\Gamma).$$

We define

$$\widetilde{Z} = \begin{cases} Z & \text{in } \mathrm{Out}(\Gamma) := \mathbb{C} \backslash \mathrm{Int}(\Gamma), \\ ZD & \text{in } \mathrm{Int}(\Gamma). \end{cases} \tag{56}$$

For $\widetilde{Z}$ we have on Γ

$$\widetilde{Z}_+ = \widetilde{Z}_- D$$

and on $[-1,1]$ this $\widetilde{Z}$ satisfies

$$\widetilde{Z_+} = Z_+ D_+ = Z_- J D_+ = Z_- D_- D_-^{-1} J D_+ = \widetilde{Z}_- (D_-^{-1} J D_+).$$

If we substitute here the decomposition of J from (54), we see that

$$D_-^{-1}JD_+ = D_-^{-1}D_-\begin{pmatrix} 0 & \dfrac{1}{\sqrt{1-x^2}} \\ -\sqrt{1-x^2} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\dfrac{1}{(h\varphi^2)_+} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{1}{(h\varphi^2)_+} & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \dfrac{1}{\sqrt{1-x^2}} \\ -\sqrt{1-x^2} & 0 \end{pmatrix}.$$

Denoting the last matrix as

$$j := \begin{pmatrix} 0 & \dfrac{1}{\sqrt{1-x^2}} \\ -\sqrt{1-x^2} & 0 \end{pmatrix},$$

we conclude that the matrix function $\widetilde{Z}$ from (56) is a solution of the following Riemann–Hilbert problem

$$\begin{cases} \widetilde{Z} \in H(\overline{\mathbb{C}}\backslash\{[-1,1]\cup\Gamma\}), \quad \Gamma\subset\delta,\ [-1,1]\subset \mathrm{Int}(\Gamma), \\ \widetilde{Z}_+ = \widetilde{Z_- D} \quad \text{on } \Gamma, \\ \widetilde{Z}_+ = \widetilde{Z}_- j \quad \text{on } [-1,1], \\ \widetilde{Z}(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z\to\infty. \end{cases} \tag{57}$$

3. The next step is to consider the limiting problem for (57), i.e., without the jump on Γ, because, as we already mentioned, this jump function tends to the identity matrix as $n\to\infty$. We consider the following Riemann–Hilbert problem for the determination of the function X:

$$\begin{cases} X \in H(\overline{\mathbb{C}}\backslash[-1,1]), \\ X_+ = X_- j \quad \text{on } [-1,1], \\ X(z) = I + O\left(\frac{1}{z}\right), \quad \text{as } z\to\infty. \end{cases} \tag{58}$$

Writing the entries for the matrix jump condition

$$\begin{pmatrix} x_{11+} & x_{12+} \\ x_{21+} & x_{22+} \end{pmatrix} = \begin{pmatrix} -x_{12-}\sqrt{1-x^2} & \dfrac{x_{11-}}{\sqrt{1-x^2}} \\ -x_{22-}\sqrt{1-x^2} & \dfrac{x_{21-}}{\sqrt{1-x^2}} \end{pmatrix},$$

we see that matrix-valued Riemann–Hilbert problem (56) reduces to two scalar problems

$$\begin{cases} x_{11}, x_{12} \in H(\overline{\mathbb{C}} \backslash [-1,1]), \\ x_{11}(z) = 1 + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty, \\ x_{12} = O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty, \\ x_{11+} = x_{12-}(-\sqrt{1-x^2}) \quad \text{on } [-1,1], \\ x_{11-} = x_{12+}(\sqrt{1-x^2}) \quad \text{on } [-1,1] \end{cases} \tag{59}$$

and

$$\begin{cases} x_{21}, x_{22} \in H(\overline{\mathbb{C}} \backslash [-1,1]), \\ x_{21}(z) = O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty, \\ x_{22} = 1 + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty, \\ x_{21+} = x_{22-}(-\sqrt{1-x^2}) \quad \text{on } [-1,1], \\ x_{21-} = x_{22+}(\sqrt{1-x^2}) \quad \text{on } [-1,1]. \end{cases} \tag{60}$$

Note that, in accordance with the chosen branch of $(z^2-1)^{\frac{1}{2}}$, we have

$$\frac{1}{i}(z^2-1)^{\frac{1}{2}}_{\mp} = \mp\sqrt{1-x^2} \quad \text{on } [-1,1],$$

therefore

$$x_{11\pm} = \left(\frac{1}{i}x_{12}(z^2-1)^{\frac{1}{2}}\right)_{\mp} \quad \text{and} \quad x_{21\pm} = \left(\frac{1}{i}x_{22}(z^2-1)^{\frac{1}{2}}\right)_{\mp}.$$

Thus the function $\left(\frac{1}{i}x_{12}(z^2-1)^{\frac{1}{2}}\right)$ is the analytic continuation of the function x_{11} on the second sheet of the Riemann surface, obtained from the two extended complex planes cut along the interval $[-1,1]$ and pasted together 'cross by cross'. Analogously, the function $\left(\frac{1}{i}x_{22}(z^2-1)^{\frac{1}{2}}\right)$ is the analytic continuation of x_{21} on the second sheet of the Riemann surface. If we choose, in accordance with the normalization at infinity,

$$x_{11} \equiv 1 \quad \text{in } \mathbb{C} \backslash [-1,1],$$

then its continuation on the second sheet will again be equal to the constant function 1, which gives

$$x_{12} = \frac{i}{(z^2-1)^{\frac{1}{2}}}.$$

Checking the normalization of x_{12} at ∞, we see that x_{11} and x_{12} as above satisfy the problem in (59).

As for the problem in (60), we have to find the function x_{21} having a zero at infinity, and the analytic continuation of x_{21} on the second sheet of the Riemann surface which needs to have a pole at infinity (because x_{22} is regular at ∞ and $(z^2-1)^{\frac{1}{2}}$ has a pole over there). The choice of x_{21} as

$$x_{21} = \frac{(z - (z^2 - 1)^{\frac{1}{2}})}{2i}$$

satisfies these conditions. Then

$$x_{22} = \frac{(z + (z^2 - 1)^{\frac{1}{2}})}{2(z^2 - 1)^{\frac{1}{2}}}$$

and the verification of the normalization of x_{22} at infinity $x_{22} = 1 + O\left(\frac{1}{z}\right)$ shows that we have found the solution for the problem in (60). Hence

$$X = \begin{pmatrix} 1 & \frac{i}{z^2-1} \\ \frac{(z-(z^2-1))^{\frac{1}{2}}}{2i} & \frac{(z+(z^2-1))^{\frac{1}{2}}}{2(z^2-1)^{\frac{1}{2}}} \end{pmatrix} \tag{61}$$

is a solution for problem (58).

4. Finally, we have to show that the solution of the Riemann problem with a jump on Γ close to the identity (see (56)), which presents orthogonal polynomials and functions of the second kind (see (55),(56) and (52)), tends as $n \to \infty$ to the solution of the problem (58) without the jump on Γ, which is the matrix function (61). To do this we define the function

$$\mathfrak{I} := \widetilde{Z} X^{-1}. \tag{62}$$

The matrix $\mathfrak{I}$ has a jump on Γ

$$\mathfrak{I}_+ = \widetilde{Z}_- D X^{-1} = \widetilde{Z}_- X^{-1} (X D X^{-1}),$$

so that

$$\mathfrak{I}_+ = \mathfrak{I}_- \widetilde{D}, \quad \widetilde{D} = X D X^{-1} \tag{63}$$

and on $[-1, 1]$ we have

$$\mathfrak{I}_+ = \widetilde{Z}_- j (X_- j)^{-1} = \widetilde{Z}_- j j^{-1} X_-^{-1} = \mathfrak{I}_-.$$

Thus the function $\mathfrak{I}$ satisfies the following Riemann–Hilbert problem (note that there is no jump on $[-1, 1]$):

$$\begin{cases} \mathfrak{I} \in H(\mathbb{C} \backslash \Gamma), \\ \mathfrak{I}_+ = \mathfrak{I}_- \widetilde{D} \quad \text{on } \Gamma, \\ \mathfrak{I}(z) = I + O\left(\frac{1}{z}\right) \quad \text{as } z \to \infty \end{cases} \tag{64}$$

and for the jump matrix $\widetilde{D}$ on Γ we have

$$\widetilde{D} = I + O(q^n) \quad \text{on } \Gamma,$$

since the entries of X and X^{-1} in (63) do not depend on n and for D the asymptotics on Γ follows from the representation of φ in (47) substituted in (55).

To finish the proof we apply to function (62) the following well-known lemma.

Lemma 1. *Suppose that the jump matrix $\widetilde{D}_n$ for the Riemann problem* (64) *is analytic in the domain $\mathcal{A}$ containing the contour Γ*

$$\widetilde{D_n} \in H(\mathcal{A}), \quad \Gamma \subset \mathcal{A}, \tag{65}$$

and satisfies

$$\widetilde{D}_n = I + \varepsilon_n, \tag{66}$$

where $\varepsilon_n \to 0$ uniformly on compact subsets of $\mathcal{A}$ as $n \to \infty$. Then

$$\mathfrak{I} = I + O(\varepsilon_n). \tag{67}$$

A proof of the lemma, with a weaker condition on the jump than (65), can be found in [3]. We note that for the case of scalar $\mathfrak{I}$, the proof of the lemma is rather trivial by considering a problem for $\log(\mathfrak{I})$ for which we can write a solution by means of the Cauchy integral of $\log(\widetilde{D})$. However, the function log of a matrix is not properly defined so that another approach is required for proving the lemma in the matrix case. This will be done in the next section.

Finally, substituting in (67) expressions (62), (58), (56), (55) and (52) we get the desired asymptotic formulas.

The theorem is proved.

5.5. *Proof of the lemma*

Assuming the more restrictive condition (65) on the jump matrix in (64) rather than the one stated in [3], we have an opportunity to give a more elementary proof of Lemma 1 than the proof in [3].

Substituting (66) in the jump condition of (64), we have the following Riemann problem for the matrix function

$$\begin{cases} \mathfrak{I} \in H(\overline{\mathbb{C}} \setminus \Gamma), \\ \mathfrak{I}_+ = \mathfrak{I}_- + \varepsilon_n \mathfrak{I}_- \quad \text{on } \Gamma, \\ \mathfrak{I}(\infty) = I. \end{cases}$$

Applying Cauchy's integral formula (see (A.1) and (A.2)) for each entry of $\mathfrak{I}$, we have

$$\mathfrak{I}(z) = \frac{1}{2\pi i} \int_\Gamma (\varepsilon_n \mathfrak{I}_-)(\xi) \frac{d\xi}{\xi - z} + I, \quad z \in \overline{\mathbb{C}} \setminus \Gamma. \tag{68}$$

Then, because of (65), we can deform Γ to Γ' which is lying inside the domain $\mathcal{A} \cap \mathrm{Out}(\Gamma)$

$$\mathfrak{I}(z) = \frac{1}{2\pi i} \int_{\Gamma'} (\varepsilon_n \mathfrak{I}_-)(\xi) \frac{d\xi}{\xi - z} + I, \quad z \in \overline{\mathbb{C}} \setminus \widetilde{\mathcal{A}}, \ \partial \widetilde{\mathcal{A}} = \Gamma \cup \Gamma' \tag{69}$$

and now the integral is not singular anymore for $z \in \Gamma$, and both sides of (69) have a limit when $z \to \Gamma_+$. Hence

$$\mathfrak{I}_- \widetilde{D} = \mathfrak{I}_+ = \int_{\Gamma'} (\varepsilon_n \mathfrak{I}_-)(\xi) \frac{d\xi}{\xi - z} + I, \quad z \in \Gamma. \tag{70}$$

Let $z_0 \in \Gamma$ be such that

$$||\mathfrak{I}_-(z_0)|| = \max_{\Gamma} ||\mathfrak{I}_-(z)|| =: M_n,$$

then from (70) we have

$$M_n \leqslant M_n \ \varepsilon_n \, \mathrm{const}_{\Gamma, \Gamma'} + 1$$

and therefore

$$M_n \leqslant \frac{1}{1 - O(\varepsilon_n)}$$

and substituting this estimate in (68), we arrive at (67).

Appendix A. The Cauchy residue theorem on a Riemann surface and the solution of the Riemann problem

A.1. Representation of piecewise analytic functions in $\overline{\mathbb{C}}$ by Cauchy's integral formula

Let us consider the piecewise analytic function f:

$$f(z) := \begin{cases} f_2(z), & z \in \mathcal{D}, \\ f_1(z), & z \in \overline{\mathbb{C}} \setminus \overline{\mathcal{D}}, \end{cases}$$

where f_1, f_2 are holomorphic functions in the closed domains of their definition

$$f_1 \in H(\overline{\mathbb{C}} \setminus \mathcal{D}), \quad f_2 \in H(\overline{\mathcal{D}}).$$

We assume that $\mathcal{D}$ is bounded, with piecewise smooth boundary $\partial \mathcal{D}$ which coincides (as a set) with $\partial(\overline{\mathbb{C}} \setminus \overline{\mathcal{D}})$. The Cauchy integral formula allows us to write f as

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathcal{D}} (f_2(\xi) - f_1(\xi)) \frac{d\xi}{\xi - z} + f_1(\infty).$$

Indeed, the integral on the right-hand side is

$$\frac{1}{2\pi i} \int_{\partial \mathcal{D}} f_2(\xi) \frac{d\xi}{\xi - z} + \frac{1}{2\pi i} \int_{\partial(\overline{\mathbb{C}} \setminus \mathcal{D})} f_1(\xi) \frac{d\xi}{\xi - z} + f_1(\infty)$$
$$= \begin{cases} f_2(z) - f_1(\infty) + f_1(\infty), & z \in \mathcal{D}, \\ 0 + f_1(z) - f_1(\infty) + f_1(\infty), & z \in \overline{\mathbb{C}} \setminus \mathcal{D}. \end{cases}$$

Here we use a version of the Cauchy theorem for a domain Ω (with $\infty \in \Omega$)

$$F \in H(\bar{\Omega}) \Rightarrow \frac{1}{2\pi i} \int_{\partial\Omega} F(\xi)\, d\xi = \operatorname*{res}_{\infty} F = -c_{-1}, \quad F(z) = F(\infty) + \frac{c_{-1}}{z} + \cdots .$$

Thus, if we introduce a jump function

$$j(\xi) := f_2(\xi) - f_1(\xi) =: f_+(\xi) - f_-(\xi), \quad \xi \in \partial\mathcal{D} =: \Gamma,$$

then the integral

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} j(\xi) \frac{d\xi}{\xi - z} + C \tag{A.1}$$

gives a solution to the following boundary value problem (BVP) for the piecewise analytic function f (with continuous boundary values f_+ and f_-):

$$\text{Find } f \text{ such that } \begin{cases} \text{(a) } f \in H(\overline{\mathbb{C}} \backslash \Gamma), \\ \text{(b) } (f_+ - f_-)|_{\Gamma} = j \in H(\Gamma), \\ \text{(c) } f(\infty) = C. \end{cases} \tag{A.2}$$

This is a BVP which is usually called the *non homogeneous Riemann problem* or '*jump*' *problem*.

Remark A.1. If $j \in H(\delta)$, where δ is such that $\Gamma \subset \delta$ and if $\Gamma' \subset \delta$, then for every $z \in \overline{\mathbb{C}} \backslash \widetilde{\delta}$, with $\widetilde{\delta} \subset \delta$ and $\partial\widetilde{\delta} = \Gamma \bigcup \Gamma'$, the solution for (A.2) can be represented as

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma'} j(\xi) \frac{d\xi}{\xi - z} + C, \quad z \in \overline{\mathbb{C}} \backslash \widetilde{\delta}. \tag{A.3}$$

Note that for $z \in \widetilde{\delta}$, the right-hand side of (A.3) does not represent the solution of (A.2).

For further insight it would be useful also to understand the Cauchy integral formula as a corollary of the more general Cauchy residue theorem. Indeed, for $f \in H(\overline{\mathcal{D}})$, with $\mathcal{D} \subset \mathbb{C}$, the formula

$$\frac{1}{2\pi i} \int_{\partial\mathcal{D}} \frac{f(\xi)}{\xi - z}\, d\xi = \operatorname*{res}_{z} \left[\frac{f(\xi)}{\xi - z} \right] = f(z)$$

is a corollary of

$$\frac{1}{2\pi i} \int_{\partial\mathcal{D}} f(\xi) w(\xi)\, d\xi = \sum_j f(z_j) \operatorname*{res}_{z_j} w = \sum_j \operatorname*{res}_{z_j} fw,$$

where $w \in \mathcal{M}$ is a meromorphic function in $\overline{\mathcal{D}}$, i.e., $w \in H(\overline{\mathcal{D}} \backslash \{z_j\})$, with $z_j \in \mathcal{D}$. The expression $h(\xi)\, d\xi$ is usually called a *meromorphic differential* in $\mathcal{D} \subset \mathbb{C}$. Thus the Cauchy differential

$$\left(\frac{1}{\xi - z} \right) d\xi$$

is an example of a meromorphic differential in $\mathbb{C}$.

A.2. Riemann surfaces, meromorphic differentials and Cauchy's theorem on a Riemann surface

A mysterious fact: why is an integral around infinity, for a function holomorphic at infinity, not equal to zero and why do we have to count the contribution of the function at ∞ despite of its holomorphicity there? This fact has an explanation in terms of the Riemann surfaces and meromorphic differentials on them.

We start with an explanation of what a *Riemann surface* means. First of all it is a *surface*, i.e., a two-dimensional topological manifold, which means that the surface $\mathcal{R}$ is locally a result of a continuous deformation of the extended complex plane (or a piece of the plane). Moreover, there is a correspondence between the points $\xi \in \mathcal{R}$ on the surface and the points $\tau \in \overline{\mathbb{C}}$ of the plane which is established by an open continuous mapping $\tau : \mathcal{R} \to \overline{\mathbb{C}}$. The mapping $\tau(\xi)$ is locally one to one everywhere except for a discrete set of points, which are called *branch* points of $\mathcal{R}$. All the other points of $\mathcal{R}$ are called *regular* points. This mapping is called a *projection* of $\mathcal{R}$.

Next, to become a Riemann surface, the surface $\mathcal{R}$ needs to have a complex structure. To explain what this means, we recall the notion of a *holomorphic* function on $\mathcal{R}$. In a neighborhood of a regular point $\xi_0 \in \mathcal{R}$ the projection $\tau(\xi)$ is invertible, therefore the mapping $\xi(\tau)$ is defined in a neighborhood of the point $\tau(\xi_0) \in \overline{\mathbb{C}}$. A function f is called holomorphic in a neighborhood of a regular point ξ_0 on $\mathcal{R}$ if the function $f(\xi(\tau))$ is holomorphic in the neighborhood of $\tau(\xi_0)$ on $\overline{\mathbb{C}}$. A function f is called holomorphic in a neighborhood U of a branch point $\xi_0 \in \mathcal{R}$ if f is continuous in U and f is holomorphic in $U \backslash \xi_0$. A pair $\mathcal{K} = (U, \tau(\zeta))$ is called a *holomorphic chart* of the surface $\mathcal{R}$ if

- U is a domain in $\mathcal{R}$,
- $\tau(\zeta)$ is holomorphic in U and maps U to the plane disk $u = \{|\tau| < r\} \in \mathbb{C}$, establishing a one to one correspondence.

In practice, Riemann surfaces are presented as a union of holomorphic charts together with formulas of transformation from one local coordinate to another, and

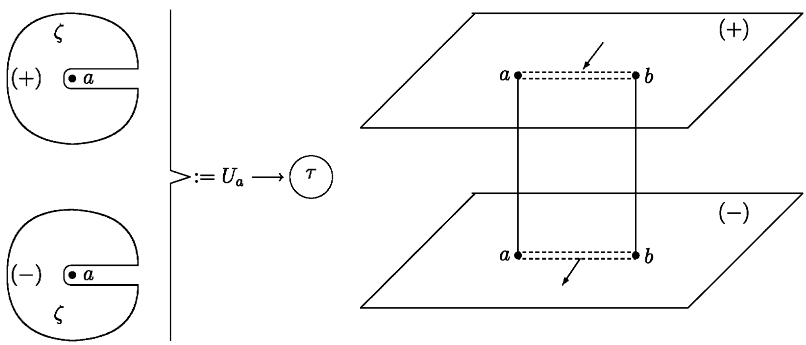

Fig. 4. A two-sheeted Riemann surface.

it is convenient to consider a function $f(\zeta)$ on $\mathcal{R}$ locally as a function $f(\zeta(\tau))$ on $\mathbb{C}$ using the local variables $\tau \in u$.

Example A. The extended complex plane $\mathcal{R} = \overline{\mathbb{C}}$ is considered as a Riemann surface with an atlas of the following holomorphic charts: $\mathcal{K}_1 = (\{|\zeta| < R\},\ \tau(\zeta) = \zeta)$, $\mathcal{K}_2 = (\{|\zeta| > r\},\ \tau(\zeta) = \frac{1}{\zeta})$, where $R \geqslant r$.

Example B. A two-sheeted Riemann surface (see Fig. 4 on the right) is a union of the following holomorphic charts:

$$\mathcal{K}_a = \left(U_a, \begin{cases} (\zeta - a)^{\frac{1}{2}} = \tau \\ \zeta = \tau^2 + a \end{cases} \right),$$

$$\mathcal{K}_b = \left(U_b, \begin{cases} (\zeta - b)^{\frac{1}{2}} = \tau \\ \zeta = \tau^2 + b \end{cases} \right), \tag{A.4}$$

$$\mathcal{K}_j = (U_j : \{a, b, \infty^{(+)}, \infty^{(-)}\} \notin U_j,\ \tau(\zeta) = \zeta), \ldots$$

$$\mathcal{K}_{\infty^{(+)}} = \left(\{|\zeta^{(+)}| > r\}, \zeta(\tau) = \frac{1}{\tau} \right), \quad \mathcal{K}_{\infty^{(-)}} = \left(\{|\zeta^{(-)}| > r\}, \zeta(\tau) = \frac{1}{\tau} \right).$$

Now a function $f(\zeta)$ on $\mathcal{R}$ can be understood locally in the domain of the chart $(U, \tau(\zeta))$ as a function of the local variable τ and its holomorphicity (meromorphicity) in U means that the function $f(\zeta(\tau))$ is holomorphic (meromorphic) in $u = \{|\tau| < r\}$.

In the same way, using a local variable, we define *meromorphic differentials on* $\mathcal{R}$ as $d\Omega(\zeta) := \omega(\zeta) d\zeta$, where $\omega(\zeta)$ is a meromorphic function on $\mathcal{R}$. For correctness of this definition, i.e., to be independent from the choice of local coordinates, it is necessary (and sufficient) to satisfy a condition of correspondence: $d\Omega_i(\zeta_i) = d\Omega_j(\zeta_j)$ in the intersection of U_i and U_j. A *singularity* of the differential $d\Omega = \omega d\zeta$ in the chart $(U, \tau(\zeta))$ is a point $\zeta_0 \in U$, $\zeta_0 = \zeta(\tau_0)$, where τ_0 is a singularity of the function (in the variable τ) $w(\zeta(\tau))\zeta'(\tau)$. The *residue* of the differential $d\Omega = \omega d\zeta$ at the point $\zeta_0 = \zeta(\tau_0)$ is

$$\operatorname*{res}_{\zeta_0} d\Omega = c_{-1},$$

where c_{-1} is the coefficient of $(\tau - \tau_0)^{-1}$ in the Laurent expansion $w(\zeta(\tau))\zeta'(\tau) = \sum_{j=-\infty}^{\infty} c_j (\tau - \tau_0)^j$. Note that, because of the holomorphic correspondence of the charts, the position of the singularity ζ_0 of $d\Omega(\zeta)$ and its residue $\operatorname{res}_{\zeta_0} d\Omega$ do not depend on the choice of the chart covering ζ_0.

Now we are able to state Cauchy's theorem on a Riemann surface. Let $f \in H(\bar{U})$, $U \subset \mathcal{R}$ ($\mathcal{R}$ is a compact Riemann surface) and $d\Omega$ be a meromorphic differential in U. Then

$$\frac{1}{2\pi i}\int_{\partial U} f(\zeta)\, d\Omega(\zeta) = \sum_j \operatorname*{res}_{\zeta_j}[f\, d\Omega], \tag{A.5}$$

where $\{\zeta_j\}$ are the singularities of $d\Omega$ in U.

A.3. The Riemann surface $\mathcal{R} = \overline{\mathbb{C}}$. Examples of meromorphic differentials on $\mathcal{R} = \overline{\mathbb{C}}$, and normalization of the Riemann problem at an arbitrary point

Here we consider differentials on $\mathcal{R} = \overline{\mathbb{C}}$ (see Example A from the previous section).

Example A1. $d\Omega(\zeta) = d\zeta$.

In the chart $\mathcal{K}_1 = (\{|\zeta| < R|\}, \zeta(\tau) = \tau)$ we have $\zeta'(\tau) = 1$, hence $d\zeta$ has no singularities for $|\zeta| < R$.

In the chart $\mathcal{K}_2 = (\{|\zeta| > r\}, \zeta(\tau) = \frac{1}{\tau})$ we have $\zeta'(\tau) = -\frac{1}{\tau^2}$, hence $d\zeta$ has a pole of second order at $\zeta = \infty$.

Thus, in accordance with (A.5), the integral for a function f, which is holomorphic at ∞,

$$f(z) = c_0 + \frac{c_{-1}}{z} + \cdots$$

is given by

$$\int_{\partial(|\zeta|>r)} f(\zeta)\, d\zeta = \operatorname*{res}_{\zeta=\infty}[f\, d\zeta] = \operatorname*{res}_{\tau=0}\left[\left(-\frac{1}{\tau^2}\right)(c_0 + c_{-1}\tau + \dots)\right] = -c_{-1}.$$

This explains the usual convention concerning integration around infinity (see Section 2.1).

Example A2. Consider the meromorphic differential $d\Omega(\zeta) = \frac{d\zeta}{\zeta - z}$.

1. In $\mathcal{K}_1$ we have $w(\zeta(\tau))\zeta'(\tau) = \frac{1}{\tau - z}$, hence $\zeta = z$ is a pole with $\operatorname{res}_{\zeta=z} d\Omega(\zeta) = 1$.
2. In $\mathcal{K}_2$ we have $w(\zeta(\tau))\zeta'(\tau) = -\frac{1}{\tau(1-\tau z)}$, hence

$$\begin{cases} \zeta = z \text{ is a pole}, & \operatorname{res}_{\zeta=z} d\Omega(\zeta) = 1, \\ \zeta = \infty \text{ is a pole}, & \operatorname{res}_{\zeta=\infty} d\Omega(\zeta) = -1. \end{cases}$$

Thus, the meromorphic Cauchy differential on the Riemann surface $\overline{\mathbb{C}}$ has two poles $\zeta = z$ and $\zeta = \infty$ with

$$\operatorname{res} d\Omega = \begin{cases} +1 & \text{at } z, \\ -1 & \text{at } \infty. \end{cases}$$

To indicate the singularities and their residues we will use the notation

$$d\Omega(\zeta) = d\Omega(\zeta; z, \infty) = d\Omega(\zeta; z_{(+1)}, \infty_{(-1)}).$$

By means of the Cauchy differential we can construct the differential

$$d\Omega(\zeta; z_{(1)}, w_{(-1)}) = \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right) d\zeta = \frac{z - w}{(\zeta - w)(\zeta - z)} d\zeta.$$

Similar to the Cauchy differential in (A.1), this differential can be used for the solution of the Riemann problem with normalization at any point $w \in \overline{\mathbb{C}}$:

$$\text{Find } f(z) \text{ such that } \begin{cases} \text{(a) } f \in H(\overline{\mathbb{C}} \backslash \Gamma), \\ \text{(b) } (f_+ - f_-)|_\Gamma = j \in H(\Gamma), \\ \text{(c) } f(w) = C_w, \end{cases}$$

$$\Rightarrow f(z) = \frac{1}{2\pi i} \int_\Gamma j(\zeta)\, d\Omega(\zeta; z, w) + C_w. \tag{A.6}$$

A.4. The Riemann problem on $\mathcal{R}$. Examples of meromorphic differentials on the two-sheeted Riemann surface

The last example of the Riemann problem on $\overline{\mathbb{C}}$ (see (A.6)) can be extended to an arbitrary compact Riemann surface $\mathcal{R}$ and oriented, piecewise smooth curve Γ on $\mathcal{R}$:

$$\text{Find } f(z) \text{ such that } \begin{cases} \text{(a) } f \in H(\mathcal{R} \backslash \Gamma), \\ \text{(b) } (f_+ - f_-)|_\Gamma = j \in H(\Gamma), \\ \text{(c) } f(w) = C_w, \ \ w \in \mathcal{R} \backslash \Gamma. \end{cases} \tag{A.7}$$

Here f_+ and f_- are the uniformly bounded limit values of f on Γ from the "positive" and "negative" side with respect to the orientation of Γ. By analogy with (A.6), from Cauchy's theorem on $\mathcal{R}$ it follows: if $j(\zeta)\, d\Omega(\zeta; z, w)$ has no singularities on Γ, then

$$f(z) = \frac{1}{2\pi i} \int_\Gamma j(\zeta)\, d\Omega(\zeta; z, w) + C_w. \tag{A.8}$$

Note the *uniqueness of the solution* of (A.7): If $\mathcal{R}$ is compact, then the solution of the Riemann problem (A.7) is unique.

Let f and $\widetilde{f}$ be the solutions of (A.7), then for $(f - \widetilde{f})$ we have $j = 0$, therefore $(f - \widetilde{f}) \in H(\mathcal{R})$ and if $\mathcal{R}$ is compact, then, by the maximum principle, $(f - \widetilde{f})$ is constant. Moreover, by (c) we see that this constant is zero.

Now we present some important meromorphic differentials on the two-sheeted Riemann surface (see Example B , Fig. 4, we put $a = -1$, $b = 1$).

Example B1. We will use the notation $z^{(+)}$ and $z^{(-)}$ for the points belonging to the "opposite" sheets, but having the same projection $z \in \overline{\mathbb{C}}$.

We have

$$d\Omega(\zeta) := \frac{\sqrt{z^2-1}}{\sqrt{\zeta^2-1}}\frac{1}{\zeta - z}\,d\zeta = d\Omega(\zeta; z^{(+)}_{(1)}, z^{(-)}_{(-1)}). \tag{A.9}$$

is a meromorphic differential on $\mathcal{R}$ with singularities described by the right-hand side of (A.9).

Proof. It is evident that $\zeta = z^{(+)}$ and $\zeta = z^{(-)}$ are the poles of $d\Omega$ with residue $+1$ and -1, respectively. Next,

$$K_{+1} : \begin{cases} \sqrt{\zeta - 1} = \tau \\ \zeta = \tau^2 + 1 \end{cases}$$

$$\Rightarrow w(\zeta(\tau))\zeta'(\tau) = \frac{2\tau\sqrt{z^2-1}}{\tau\sqrt{\tau^2+2}(\tau^2+1-z)}$$

$$\Rightarrow \text{no singularity at } \zeta = 1.$$

The same at $\zeta = -1$. There is no singularity at $\zeta = \infty^{(\pm)}$: because of

$$\mathcal{K}_{\infty^{(\pm)}} : \zeta = \frac{1}{\tau}, \quad w(\zeta(\tau))\zeta'(\tau) = \frac{\tau^2\sqrt{z^2-1}}{\sqrt{1-\tau^2}(1-\tau z)}\left(-\frac{1}{\tau^2}\right).$$

Example B2.

$$d\Omega(\zeta) := \frac{1}{2}\frac{\sqrt{\zeta^2-1}+\sqrt{z^2-1}}{(\zeta - z)\sqrt{\zeta^2-1}}\,d\zeta = d\Omega(\zeta; z_{(1)}, \infty^{(+)}_{(-\frac{1}{2})}, \infty^{(-)}_{(-\frac{1}{2})}). \tag{A.10}$$

Proof. There is a singularity at z, with $\operatorname{res}_z d\Omega = 1$, but there is no singularity at $\widetilde{z}$. There are no singularities at $\zeta = \pm 1$. Indeed (take $+1$ for example)

$$\mathcal{K}_{+1} \Rightarrow w(\zeta(\tau))\zeta'(\tau) = \frac{\tau\sqrt{\tau^2+2}+\sqrt{z^2-1}}{2(\tau^2+1-z)\tau\sqrt{\tau^2+2}}2\tau.$$

At $\infty^{(\pm)}$ we have

$$w(\zeta(\tau))\zeta'(\tau) = \frac{1}{2}\frac{\sqrt{1-\tau^2}+\tau\sqrt{z^2-1}}{(1-\tau z)\sqrt{1-\tau^2}}\tau\left(-\frac{1}{\tau^2}\right),$$

therefore

$$\operatorname*{res}_{\infty^{(\pm)}} d\Omega = -\frac{1}{2}.$$

From (A.10) we obtain the expression for

$$d\Omega(\zeta; z_{(1)}, w_{(-1)}) = d\Omega\left(\zeta; z_{(1)}, \infty^{(+)}_{(-\frac{1}{2})}, \infty^{(-)}_{(-\frac{1}{2})}\right) - d\Omega\left(\zeta; w_{(1)}, \infty^{(+)}_{(-\frac{1}{2})}, \infty^{(-)}_{(-\frac{1}{2})}\right).$$

Example B3. (Green's differential).

$$dG(\zeta) := \frac{d\zeta}{\sqrt{\zeta^2 - 1}} = d\Omega\left(\zeta; \infty^{(+)}_{(-1)}, \infty^{(-)}_{(+1)}\right). \tag{A.11}$$

Proof.

$$\mathcal{K}_{+1} : \zeta = \tau^2 + 1 \Rightarrow w(\zeta(\tau))\zeta'(\tau) = \frac{2\tau}{\tau\sqrt{\tau^2 + 2}}.$$

$$\mathcal{K}_{\infty^+} : \zeta = \frac{1}{\tau} \Rightarrow w(\zeta(\tau))\zeta'(\tau) = \frac{\tau}{\sqrt{1 - \tau^2}}\left(-\frac{1}{\tau^2}\right) \Rightarrow \operatorname*{res}_{\infty^+} dG = -1.$$

From (A.10) and Green's differential (A.11) we obtain the expression for the differential

$$d\Omega(\zeta; z_{(1)}, \infty^{(-)}_{(-1)}) = d\Omega\left(\zeta; z_{(1)}, \infty^{(+)}_{(-\frac{1}{2})}, \infty^{(-)}_{(-\frac{1}{2})}\right) - \frac{1}{2}dG. \tag{A.12}$$

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